Lecture 13 (Graph Isomorphism)

Definition 1 Let $G = (V, E)$ be a graph and let $u, v \in V$.

1. A u-v walk W is a finite sequence of vertices $(u = v_1, v_2, \ldots, v_n = v)$ such that $v_i \sim v_{i+1}$ for all $i = 1, 2, \ldots, n-1$.
2. The length of a walk $W = (u = v_1, v_2, \ldots, v_n = v)$ is the number of edges in W, that is n.
3. A walk $W = (v_1, v_2, \ldots, v_n)$ is called a trail if all the edges are distinct.
4. A walk $W = (u = v_1, v_2, \ldots, v_n = v)$ is called a path P if all its vertices, and hence the edges, are distinct. The vertices u and v are called end vertices and the remaining vertices are called internal vertices of P.
5. A walk (trail, path) $W = (u = v_1, v_2, \ldots, v_n = v)$ is called closed if $u = v$.
6. A closed path is called a cycle. Thus, in a simple graph, a cycle has length at least 3.
7. The graph G is called connected if for any vertices $u, v \in V$, there is a u-v path.

Definition 2 A graph which is not connected is called disconnected. If G is disconnected, then a maximal connected subgraph of G is called a component or connected component of G.

A graph G is called a null graph if it has no edges.

Proposition 1 *Let u and v be distinct vertices in a graph G. Let $W = [u = u_1, \ldots, u_k = v]$ be a walk in G. Then W contains a $u - v$ path.*

Let $G = (V, E)$ be a graph. Then we define some terms associated to G.

Definition 3 Let $G = (V, E)$ be a graph.

1. The distance $d(u, v)$ between two vertices $u, v \in V$, $u \neq v$, is the shortest length of $u - v$ path in G. If there is no such path then $d(u, v) = \infty$.
2. The diameter $\text{diam}(G)$ of G is the greatest distance between any two vertices in G. The diameter of a disconnected graph is ∞.
3. The eccentricity $e(u)$ of a vertex u is defined as $e(u) = \max\{d(u, v) \mid v \in V\}$. If G is disconnected then eccentricity of each vertex is ∞.

4. The radius $r(G)$ of G is defined as $r(G) = \min\{e(v) \,|\, v \in V\}$. Since the eccentricity of every vertex in a disconnected graph is infinity, hence the radius of a disconnected graph will be infinity.

5. A point $u \in V$ is called a center point of (G) if $e(u) = r(G)$. A collection of all the center points is called the center of G and is denoted as $C(G)$.

6. The length of longest cycle in G is called the circumference of G.

7. The length of shortest cycle in G is called the girth of G and is denoted as $g(G)$. If G has no cycle then $g(G) = \infty$.

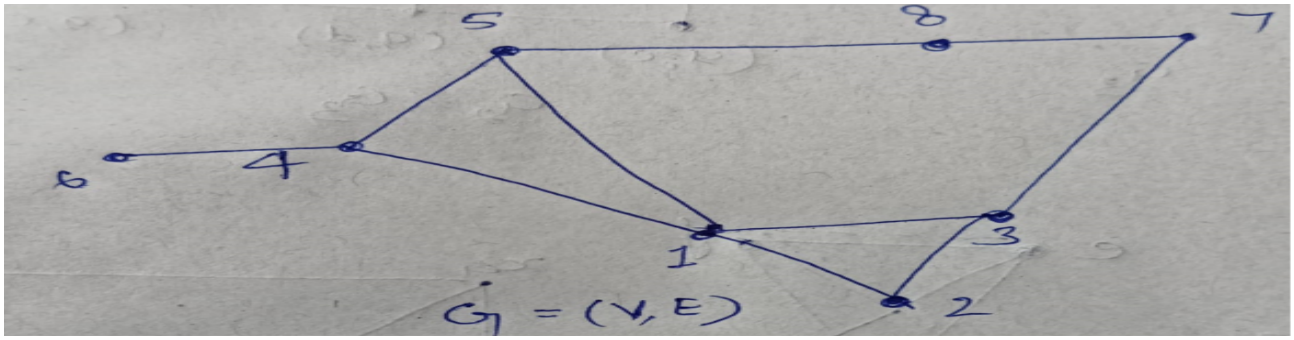
8. A complete subgraph of G is called a clique of G. The maximum order of a clique is called the clique number of G and is denoted as $\omega(G)$.

Proposition 2 *Let $G = (V, E)$ be a non-null graph. Then G is disconnected if and only if the vertex set V can be partitioned into two parts, say V_1, V_2, such that if $e = uv \in E$ then either both $u, v \in V_1$ or both $u, v \in V_2$.*

Proof: ($\Longleftarrow$) Suppose V can be partitioned into two parts V_1 and V_2, satisfying the stated condition in the proposition. Since, V_1 and V_2 are non-empty, let $u \in V_1$ and $v \in V_2$. Let P be path joining u and v. There there is an edge $e = xy$ such that $x \in V_1$ and $y \in V_2$. This contradicts the assumption that either both $x, y \in V_1$ or $x, y \in V_2$. Hence no such P exists.

($\Longrightarrow$) Let us assume that G is disconnected. Now fix a vertex $u \in V$ and consider $V_1 = \{x \in V \,|\, d(u, v) = \infty\}$. Since G is disconnected, V_1 is a proper subset of V and hence the set $V_2 = V \setminus V_1$ is non-empty subset of V. Clearly, V_1 and V_2 give a partition of V fulfilling the given condition. This completes the proof.

Example: Consider the following graph G. For the vertex 1, $d(1, 2) = d(1, 3) = d(1, 4) = d(1, 5) = 1$, $d(1, 6) = d(1, 7) = d(1, 8) = 2$, therefore $e(1) = 2$. Similarly, we see $e(2) = e(3) = e(4) = 3$, $e(5) = 2$, $e(6) = e(7) = 4$, and $e(8) = 3$. This gives $r(G) = 2$, $diam(G) = 4$, and $C(G) = \{5\}$. Also, we see $\omega(G) = 3$. The girth and circumference of G are 3 (consider $C_1 = (1, 2, 3)$) and 7 (consider the cycle $C_2 = (1, 2, 3, 7, 8, 5, 4)$ respectively).



Proposition 3 *Let G be a graph with $|E(G)| \geq 2$ and $\deg(v) \geq 2$, for each vertex except one, say v_1. Then, G has a cycle.*

Proof: Let $P = (v_1, v_2, \ldots, v_k)$ be the largest path. As $\deg(v_k) \geq 2$, v_k must be adjacent to some vertex $v_2, v_3, \ldots, v_{k-2}$, otherwise we can extend the path to larger path. Choose $i \geq 2$ such that v_i is adjacent to v_k. Then $(v_i, v_{i+1}, \ldots, v_k)$ is a cycle.

Proposition 4 *Let P and Q be two different u-v paths in G. Then, $P \cup Q$ contains a cycle.*

Proof: Since $P \neq Q$, there is a vertex $w \in P \cap Q$ s.t. edges $e = wx$ and $e' = wy$ are distinct and $e \in P$, $e' \in Q$. Choose w so that this is the first vertex with this property after the initial vertex u (it is possible that $w = u$). Now let z be the vertex nearest to w such that $z \in P \cap Q$ and $z \neq w$ (it is possible that $z = v$). Then $(w, x, \ldots, z, \ldots, y, w)$ is the required cycle.

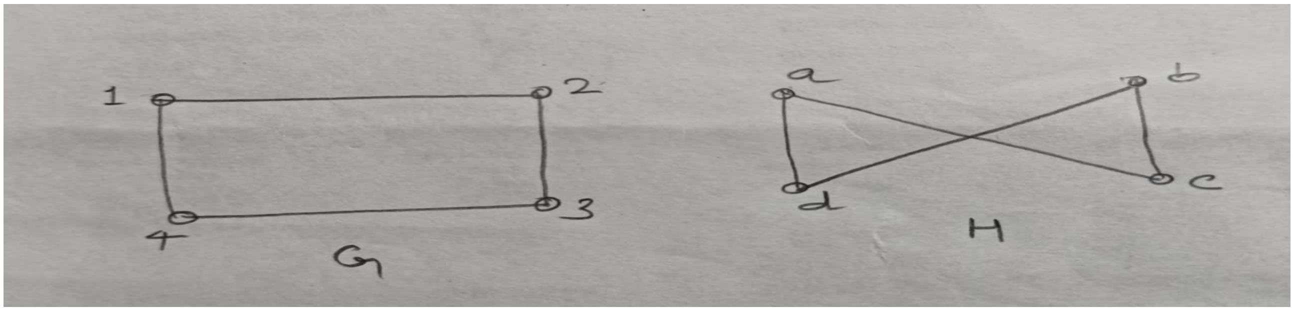
Proposition 5 *Every graph G containing a cycle satisfies $g(G) \leq 2\, diam\,(G) + 1$.*

Proof: Let $C = (v_1, v_2, \ldots, v_k)$ be the shortest cycle with length k, i.e., $g(G) = k$, and let $diam(G) = r$. Suppose $k \geq 2r + 2$. Then consider the path $P = (v_1, v_2, \ldots, v_{r+2})$. Since the length of P is $r + 1$ and $diam(R) = r$, there is a $v_{r+2} - v_1$ path say R of length at most r. Note that P and R are different $v_1 - v_{r+2}$ paths. By above proposition, the closed $P \cup R$ of length at most $2r + 1$ contains a cycle. Hence the length of this cycle is at most $2r + 1$, a contradiction to C having the smallest length $k \geq 2r + 2$.

1 Graph Isomorphism

Definition 4 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Then G_1 is isomorphic to G_2, denoted as $G_1 \cong G_2$, if there is a bijection $f : V_1 \to V_2$ such that $u\,v \in E_1$ if and only if $f(u)\,f(v) \in E_2$. The map f is called an isomorphism. If $G_1 = G_2 = G$, then f is called an automorphism on G.

Example. Note that the following graphs G and H are isomorphic by the map $f : G \to H$ s.t. $f(1) = a$, $f(2) = c$, $f(3) = b$, and $f(4) = d$.



Remark: 1: The identity map is always an automorphism on any graph.

2: There are only two automorphisms on a path P_8. What about for P_n, $n \geq 3$?

Definition 5 A graph G is called self-complementary if $G \cong \overline{G}$.

Example: Let $G = (V, E)$ be a self complementary graph on n vertices. Then $|E| = \frac{n(n-1)}{4}$, as $|E| = |\bar{E}|$ and there are $\frac{n(n-1)}{2}$ edges in complete graph. Thus either $n = 4k$ or $n = 4k + 1$. Verify that the path $P_4 = (0, 1, 2, 3)$ and the cycle $C_5 = (a, b, c, d, e)$ are self complementary.

Definition 6 A graph invariant is a function which assigns the same value to isomorphic graphs. Observe that some of the graph invariants are: $|V|$, $|E|$, $\Delta(G)$ (maximum degree), $\delta(G)$ (minimum degree), $\omega(G)$ (clique number), $r(G)$ (radius), $e(G)$ (eccentricity).

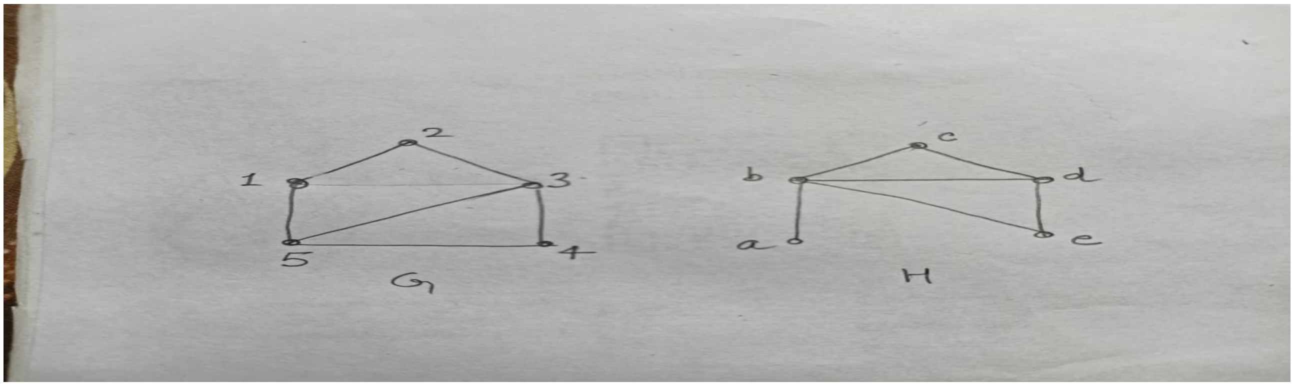
Proposition 6 *Let G and H be graphs and let $f : G \to H$ be an isomorphism. For any $v \in V(G)$, $G - v \cong H - f(v)$.*

Proof: Consider the bijection $g : V(G - v) \to V(H - f(v))$ described by $g = f_V(G - v)$.

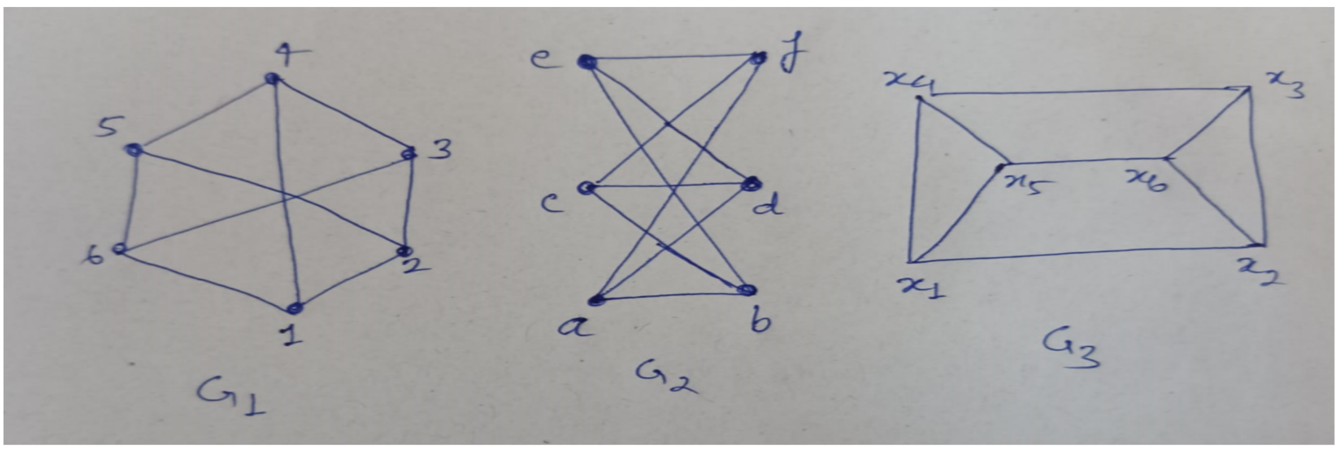
Example. Note that in the below figure, the graphs G and H are not isomorphic. Although they have same number of vertices and edges, but the vertex b in H has degree 4, while G has no vertex of degree 4.

We may also use Proposition 6 to show that they are non-isomorphic. Assume G and H are isomorphic (via an isomorphism $f : H \to G$), then by the above proposition, the subgraphs $H - b \cong G - f(b)$. However, $H - b$ has two connected components while $G - f(b)$ has only one connected component for any image of b under the f.

One may use graph invariants also. Note that the circumference of G is 5, while it is 4 for H.



Example: Consider the graphs G_1, G_2, and G_3. Observe that, G_1 is isomorphic to G_2 via $f : G_1 \to G_2$ such that $f(1) = d$, $f(2) = a$, $f(3) = b$, $f(4) = e$, $f(5) = f$, and $f(6) = c$. However, G_3 is not isomorphic to G_1 or G_2, using girth, $g(G_3) = 3$ while $g(G_1) = g(G_2) = 4$. Thus $G_1 \cong G_2 \ncong G_3$.



References:

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2: K. H. Rosen; Discrete mathematics and its applications, Tata McGraw-Hill.